On the Maximal Value of the Turbulent α-Parameter in Accretion Discs

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Abstract

In this short paper we show that making turbulence two- rather than three-dimensional may increase effective turbulent viscosity by about 40%. Dimensionless hydrodynamical viscosity parameter up to $\alpha_{max} = 0.25\mathcal{M}_t^2$ may be obtained in this approach, that is in better agreement with the observational data on non-stationary accretion than the values obtained in numerical simulations. However, the α-parameter values known from observations are still several times higher.

1 Introduction

Though α prescription (Shakura 1973; Shakura & Syunyaev 1973) proved to be quite useful for accretion disc physics, the physical mechanisms driving angular momentum transport are still not completely understood. Hydrodynamic as well as MHD turbulence and magnetic fields are generally proposed as the main viscosity sources. One of the sources of MHD turbulence is magneto-rotational instability first predicted in the classic works by Velikhov (1959) and Chandrasekhar (1960) and applied for the case of disc accretion by Balbus & Hawley (1991).

Stationary accretion disc properties depend weakly on the viscosity α-parameter itself. $T_{eff} \propto R^{-3/4}$ effective temperature profile (and hence the outcoming multicolor blackbody spectrum, see Lynden-Bell (1969)) is independent of viscosity. However, most of the physical quantities (such as density, radial velocity and the total mass of the disc) depend somehow on the viscosity parameter.

Detailed spectral fitting is capable for viscosity parameter estimates through the density of the disc atmosphere. Few attempts (such as Davis et al. (2006)) were made yet to determine α-parameter from the X-ray spectra, but these estimates are still uncertain and highly model-dependent.

Due to that reason, α-parameter is generally determined from observational data on non-stationary accretion phenomena such as dwarf nova flares and X-ray transients (Smak 1999; King et al. 2007; Suleimanov et al. 2008). The actual viscosity parameter value generally affects characteristic viscosity timescales and subsequently the shapes of optical and X-ray lightcurves. Qiao & Liu (2009) use

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transitions between the low-hard and high-soft states in X-ray binaries to estimate the actual α-parameter values in the range $0.1 \div 0.9$.

The dimensionless viscosity parameter is proven to vary in the range $0.1 \div 0.9$ for hot, fully ionized accretion discs. Numerical MHD simulations generally predict $\alpha \lesssim 0.02$ (see King et al. (2007) for review) much smaller than the values required by observations.

Slightly higher values such as $0.1 \div 0.2$ appear in hydrodynamical turbulence models such as that of Kato (1994). Turbulence is treated as a three-dimensional phenomenon. There are however reasons to consider also the two-dimensional case because vertical turbulent motions are damped by positive vertical entropy gradient (Shakura et al. 1978; Tayler 1980). Here we follow the general line of Kato (1994); Narayan et al. (1994); Kato & Yoshizawa (1997); Kato et al. (1998), but consider two-dimensional case instead of three-dimensional. In the following section we write out the general form of the Reynolds stress evolution equations for the 2D-case. In section 3 a steady-state axisymmetric solution is established. We estimate the dimensional viscosity parameter value in section 4 and discuss our results and make conclusions in sections 5 and 6.

2 Reynolds Stress

Here we define the turbulent stress tensor as $t_{ij} = \langle u_i u_j \rangle$, where $u_{i,j}$ are the fluctuating velocity components. The flow is considered incompressible, that presumably does not affect a non-divergent flow. We modify the equations by Kato & Yoshizawa (1997) in a way to account for the different dimensionality of the case considered. In our case, anisotropy tensor is naturally defined as $b_{ij} = (t_{ij} - K)/2K$, where $i, j = r, \varphi$ and $K = (t_{rr} + t_{\varphi\varphi})/2$. Dissipation rate ε should also appear without the $2/3$ miltiplier present in the three-dimensional approach. Stress tensor evolution equations may be written as follows:

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) t_{rr} = 4\Omega t_{r\varphi} + \Pi_{rr} - \varepsilon \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) t_{\varphi\varphi} = -\frac{\varkappa^2}{\Omega} t_{r\varphi} + \Pi_{\varphi\varphi} - \varepsilon \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) t_{r\varphi} = 2\Omega t_{\varphi\varphi} - \frac{\varkappa^2}{2\Omega} t_{rr} + \Pi_{r\varphi} \tag{3}$$

For the pressure-strain correlation terms Π_{ij} we adopt the general form with two dimensionless constants, used by Kato (1994), $\Pi_{ij} = -C_1 K \varkappa b_{ij} + C_2 K S_{ij}$. $C_{1,2}$ are universal dimensionless constants, and $S_{ij} = (\partial U_i/\partial x_j + \partial U_j/\partial x_i)/2$ is the rate of strain of the avaraged flow. Following Kato (1994), we use $\varkappa$ rather than Ω (Kato et al. 1998) as the characteristic frequency for local processes such as inertial waves. That assumption makes no difference for the case of Keplerian rotation when $\varkappa = \Omega$. Results may be modified by substituting $C_1\Omega/\varkappa$ instead of C_1 in all the equations. We ingore the two additional terms with coefficients C_3 and C_4 (Kato & Yoshizawa 1997). Their inclusion (for any $C_4 > 0$ and $0 < C_3 < 2$) may only decrease the $t_{r\varphi}$ stress component value.

Finally, pressure-strain correlation terms take the form:

$$\Pi_{rr} = -\frac{C_1}{2} \varkappa (t_{rr} - K) \tag{4}$$

$$\Pi_{\varphi\varphi} = -\frac{C_1}{2}\varkappa\left(t_{\varphi\varphi} - K\right) \tag{5}$$

$$\Pi_{\varphi\varphi} = -\frac{C_1}{2}\varkappa t_{r\varphi} + \frac{C_2}{2} r \frac{d\Omega}{dr} t_{r\varphi} \tag{6}$$

3 Steady-State Axisymmetric Solution

Stationary solution is easily obtained from equations (1-3) by zeroing their left-hand sides. It is instructive also to obtain an expression for the steady-case dissipation term as a sum of equations (1) and (2). Kinetic energy K is also conserved in stationary case, hence:

$$0 = \frac{\mathrm{d}K}{\mathrm{d}t} = -r\frac{\mathrm{d}\Omega}{\mathrm{d}r} t_{r\varphi} - \varepsilon \tag{7}$$

The dissipation rate may be therefore written as $\varepsilon = -rd\Omega/drt_{r\varphi}$. Note that the two pressure-strain correlation terms exactly zero after summation. After some algebra one obtains the components of the Reynolds stress tensor (normalised by $2K$) as follows:

$$\begin{aligned} \tfrac{t_{rr}}{2K} = \tfrac{1}{2}\Big(1 - \tfrac{2}{A_0}\left(1 - \tfrac{C_2}{2}\right) \times \\ \times \left(2 + \tfrac{1}{2}\tfrac{d\ln\Omega}{d\ln r}\right)\tfrac{d\ln\Omega}{d\ln r}\right) \end{aligned} \tag{8}$$

$$\begin{aligned} \tfrac{t_{\varphi\varphi}}{2K} = \tfrac{1}{2}\Big(1 - \tfrac{1}{A_0}\left(1 - \tfrac{C_2}{2}\right) \times \\ \times \left(-\tfrac{\varkappa^2}{\Omega^2} + \tfrac{d\ln\Omega}{d\ln r}\right)\tfrac{d\ln\Omega}{d\ln r}\Big) \end{aligned} \tag{9}$$

$$\frac{t_{r\varphi}}{2K} = -\frac{C_1}{4A_0}\frac{\varkappa}{\Omega}\left(1 - \frac{C_2}{2}\right)\frac{d\ln\Omega}{d\ln r}, \tag{10}$$

where:

$$A_0 = \left(\frac{C_1^2}{4} + 4\right)\frac{\varkappa^2}{\Omega^2} + \left(\frac{d\ln\Omega}{d\ln r}\right)^2 \tag{11}$$

4 Estimating Turbulent α-Parameter

Let us estimate the turbulent viscosity parameter for the two-dimensional case. We define the dimensionless α parameter as:

$$\alpha = \frac{\rho|t_{r\varphi}|}{p} = \frac{\gamma|t_{r\varphi}|}{c_s^2},$$

where c_s is the speed of sound, and $\mathcal{M}_t^2 = 2K/c_s^2$ is therefore the turbulent Mach number squared. For the general case of arbitrary $C_{1,2}$:

$$\alpha = \frac{\gamma C_1}{4A_0}\frac{\varkappa}{\Omega}\left(1 - \frac{C_2}{2}\right)\left|\frac{d\ln\Omega}{d\ln r}\right|\mathcal{M}_t^2 \tag{12}$$

As in the three-dimensional case, α-parameter has a maximum at a certain value of $(C_1)_{max}$. In the two-dimensional case $(C_1)_{max} = 5$ for Keplerian rotation. The maximal value of α is:

$$\alpha_{max} = \frac{\gamma}{8}\frac{\varkappa}{\Omega}\frac{1}{\sqrt{1+\frac{1}{4}\frac{\Omega^2}{\varkappa^2}\left(\frac{d\ln\Omega}{d\ln r}\right)^2}}\left|\frac{d\ln\Omega}{d\ln r}\right|\mathcal{M}_t^2 \tag{13}$$

Maximal α-parameter values for two- and three-dimensional case differ by about 40%. In the particular case of Keplerian rotation, $C_2 = 0$ and $\gamma = 5/3$,

$$\alpha_{max} = \begin{cases} 0.18\mathcal{M}_t^2 & \text{for 3D} \\ 0.25\mathcal{M}_t^2 & \text{for 2D} \end{cases} \tag{14}$$

The second expression is the precise upper limit for two-dimensional hydrodynamic turbulence. The difference with the three-dimensional case is that all the turbulent energy is concentrated in radial and azimuthal motions that contribute to the radial angular momentum transfer. Presence of the additional z dimension leads to decrease of radial and azimuthal motion amplitudes.

5 Discussion

The viscous α-parameter may be defined in different ways. Above we used $\alpha = \alpha_p = \rho t_{r\varphi}/p$. Alternatively, α parameter is sometimes defined using the turbulent viscosity coefficient $\nu = \alpha_\nu c_s^2/\Omega$. These two definitions of the α-parameter are related by the following expression:

$$\alpha_p = \frac{\rho t_{r\varphi}}{p} = \frac{\gamma\nu}{c_s^2} r \left|\frac{d\Omega}{dr}\right| = \gamma\left|\frac{d\ln\Omega}{d\ln r}\right|\alpha_\nu \tag{15}$$

The two parameter definitions α_p and α_ν are absolutely equivalent. For Keplerian rotation and $\gamma = 5/3$, α_p is higher by a factor 2.5. Therfore, our $(\alpha_p)_{max} = 0.25$ corresponds to $(\alpha_\nu)_{max} = 0.1$.

Vertical structure of a thin accretion disc is unfavourable for strong turbulence development in z-direction. If net entropy increases with the distance from the equatorial plane of the disc, vertical motions are expected to be damped. That is the case for standard accretion discs in gas pressure-dominated regime (Shakura et al. 1978; Tayler 1980), excluding the zones of partial ionization of hydrogen where super-adiabatic temperature gradient appears (Meyer & Meyer-Hofmeister 1982). Stable vertical stratification affects mostly the largest turbulent spatial and velocity scales, where vertical motions are effectively damped by buoyancy forces. The situation is different for horizontal turbulent motions that always have an energy influx from the averaged shear flow. Thin accretion discs are predicted to be unstable with respect to radial convective motions (Rossby wave instability, see Morozov & Hoperskov (1990); Lovelace et al. (1999); Zhuravlev & Shakura (2007)). Largest vortices of the turbulent cascade receive their energy and momentum immediately from the averaged flow. Due to that they are expected to be close to complanar with the accretion disc rotation. On the other hand, vertical motions do not interact directly with the mean flow. Because the mean flow is primarily affected by the largest eddies of the turbulent cascade, turbulent velocity field is effectively two- rather than three-dimensional, when its dynamical effect on the mean flow is considered.

Excitation mechanisms and energy balance of hydrodynamical turbulent motions in accretion discs are tightly connected with the instabilities present in the

flow. Though thin accretion discs generally fulfil the Reynolds and Høiland stability criteria, they may be unstable to non-axisymmetric perturbations in the presence of radial entropy gradient (Lovelace et al. 1999). Besides this, even in the linearly stable case there are certain ways to destabilize differentially rotating flows through non-linear instabilities, mode coupling and transient growth solutions (see Umurhan & Regev (2004) for review and references therein). Recurrent transient growth solutions may lead to large turbulent energy growth for the large Reynolds numbers characteristic for accretion discs (Afshordi et al. 2005). In short, it seems that turbulence is likely to develop even in linearly stable cases. Simulations are not consistent however in this conclusion – certain numerical simulations (Hawley et al. 1999) report stabilization of all the instability modes in shearing box simulations.

Turbulence excitation in differentially rotating fluids are being extensively studied in laboratory experiments with Taylor-Couette flows. A comprehensive explanation of the first experiments of that kind (Taylor 1936) was given by Zeldovich (1981), who also points out the similarity between rotating fluid motions and astrophysical discs. So far, laboratory experiments do not support high viscosity values in differentially rotating fluids with quasi-Keplerian profiles. Effective α values found by Ji et al. (2006) are well below $\sim 10^{-5}$ for high Reynolds numbers $\mathrm{Re} \gtrsim 10^{6}$. Taylor-Couette experiment, however, differs from astrophysical discs in some important points like radial entropy gradient and gravitational field of the central body. Resulting non-axisymmetric convective instability modes may be therefore responsible for high accretion rates in astrophysical discs. Besides this, astrophysical discs are highly supersonic flows, while Mach numbers in laboratory experiments are much lower unity.

Numerical simulations mostly confirm the existence of long-living turbulence in rotating shear flows. Large scale two-dimensional vortices, mainly of anticyclonic nature, are reported to form and survive (Bodo et al. 2007; Lithwick 2009; Shen et al. 2006), sometimes exceeding disc thickness in their spatial extent in azimuthal direction. Bodo et al. (2007) report that the azimuthal size of a typical vortex exceeds the scaleheight of the disc for low speed of sound values (or high Mach numbers of the flow), starting from about 0.05 in Keplerian velocity units. So the presence of the large-scale two-dimensional vortices relevant for angular momentum transfer is well confirmed by numerical simulations. It is not clear whether the vortices themselves are stable if the effect of the third (vertical) direction is taken into account. Lithwick (2009) claims that vortex dynamics and stability properties do not change significantly in 3D with respect to the 2D case. On the other hand, another recent paper by Lesur & Papaloizou (2009) report that elliptic 2D vortices are almost always destroyed by resonance processes with the local epicyclic oscillations.

6 Conclusions

Our main conclusion is that the turbulent viscosity α-parameter may increase by up to 40% if the turbulence is two- rather than three-dimensional. The resulting upper limit $(\alpha_p)_{max} = 0.25\mathcal{M}_t^2$ is still too low to explain the high viscosity parameter values derived from non-stationary accretion phenomena.

It is possible that the contemporary turbulence theory based either on laboratory experiments or a quasi-particle approach misses some important effects lead-

ing to more effective turbulent motion interaction. Alternatively, one may allow turbulence to be supersonic or to consider the influence of magnetic fields. The impact of the latter in controversial: as it was already mentioned in Introduction, MHD simulations predict relatively low α-parameter values.

We conclude that the observational viscosity parameter is still $2\div3$ times higher than the upper limits given by theory. The problem is yet to be solved.

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